Some Aspects of Planck Scale Quantum Optics

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Abstract

This paper considers the effects of gravitational induced uncertainty on some well-known quantum optics issues. First we will show that gravitational effects at quantum level destroy the notion of harmonic oscillations. Then it will be shown that, although it is possible(at least in principle) to have complete coherency and vanishing broadening in usual quantum optics, gravitational induced uncertainty destroys complete coherency and it is impossible to have a monochromatic ray. We will show that there is an additional wave packet broadening due to quantum gravitational effects.

PACS: 04.60.-m, 42.50.-p, 42.25.-p, 11.17.+y

Keywords: Quantum Gravity, Quantum Optics, Wave Packet Propagation, Coherent States

1 Introduction

Harmonic analysis is a primary input for a vast number of technics and approaches in quantum optics. The possible break down of this simple notion which is the essence of Fourier analysis, should result in a variety of novel implications. If one be able to show that there is no harmonic oscillation essentially, a number of technics and concepts should be re-examined. Here, we will show that, when one considers quantum effects of gravity, the very notion of harmonicity breaks down. This feature implies some new implications for the rest of quantum optics. In usual quantum optics, one can have coherent states in principle. These states are states with minimum uncertainties(maximum localization) and therefore minimum broadening when they propagates. In other words, based on Heisenberg uncertainty principle, $\Delta x \Delta p \geq \hbar$, it is possible, in principle, to have localized states, and as a result, a wave packet can propagates from one point to another point without any broadening(the so-called solitonic states). When one considers gravitational effect at quantum level, the situation differs considerably. Gravity induces uncertainty and this extra uncertainty will produce new quantum optical phenomena. As a result, although it is possible to have complete coherency and vanishing broadening in usual quantum mechanics, gravitational induced uncertainty destroys complete coherency and it is not possible to have a monochromatic ray in principle. The goal of this paper is the investigation of such a new quantum gravitational induced phenomena.
The structure of the paper is as follows: Section 2 gives an overview to Generalized Uncertainty Principle(GUP). In section 3 we will show that there is no harmonic oscillation in gravitational quantum optics. In section 4 the problem of coherent states for harmonic oscillation is discussed. We will show that due to the failure of the notion of harmonic oscillation, although there is no considerable difference in definition of coherent states relative to ordinary quantum mechanics, considering expectation values and variance of some operators, quantum gravitational arguments leads to the result that complete coherency is impossible in extreme quantum gravity regime. Section 5 considers the effect of gravitation on wave packet propagation. We will show that there is an extra broadening due to gravitational induced uncertainty. Summary and conclusions are presented in section 6.

2 Generalized Uncertainty Principle

Recently it has been indicated that measurements in quantum gravity should be governed by generalized uncertainty principle. There are some evidences from string theory[1-5], black holes Physics gedanken experiments[6,7] and loop quantum gravity[8], which leads some authors to re-examine usual uncertainty principle of Heisenberg. These evidences have origin on the quantum fluctuation of the background spacetime metric. Introduction of this idea has drown considerable attention and many authors considered various problems in the framework of generalized uncertainty principle[9-20]. Such investigations have revealed that in Planck scale a re-formulation of quantum theory is un-avoidable. This re-formulated quantum theory should incorporate gravitational effects from very beginning. In this extreme quantum level, spacetime is not commutative[21] and based on some general arguments it is possible to interpret gravity as a consequence of some unknown quantum effects[22]. As another novel consequence of such re-formulated quantum theory, constants of the nature may vary with time[23,24]. In addition, the very notion of locality and position space representation are not satisfied in Planck scale[25,26] and one has to consider Hilbert space of maximally localized states. In the same manner which uncertainty principle provides a thorough foundation for usual quantum theory, now generalized uncertainty principle(GUP) is the cornerstone of modified quantum theory. Generalized uncertainty principle leads naturally to the existence of a minimal observable length on the order of Plank length l_P. A generalized uncertainty principle can be formulated as

$$\Delta x \geq \frac{\hbar}{\Delta p} + const.G\Delta p, \tag{1}$$

which, using the minimal nature of l_P can be written as,

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha' l_p^2 \frac{\Delta p}{\hbar}. \tag{2}$$

The corresponding Heisenberg commutator now becomes,

$$[x,p] = i\hbar(1+\beta p^2). \tag{3}$$

Actually as Kempf *et al* have argued[25], one can consider more generalization such as

$$\Delta x \Delta p \geq \frac{\hbar}{2}\Big(1+\alpha(\Delta x)^2+\beta(\Delta p)^2+\gamma\Big) \tag{4}$$

and the corresponding commutator relation is

$$[x,p] = i\hbar(1+\alpha x^2+\beta p^2). \tag{5}$$

This statement shows that GUP itself has a perturbational expansion. In which follows, since we are dealing with dynamics, we consider only equation (2) or equivalently (3). The main consequence of this GUP is that measurement of position is possible only up to Plank length, l_P. So one can not setup a measurement to find more accurate particle position than Plank length. In other words, one can not probe distances less than Planck length.

3 GUP and Harmonic Oscillations

The problem of harmonic oscillation in the context of GUP first has been considered by Kempf *et al*[25]. They have found eigenvalues and eigenfunctions of harmonic oscillator in the context of GUP by direct solving of the Schrödinger equation. Then Camacho has analyzed the role that GUP can play in the quantization of electromagnetic field. He has considered electromagnetic oscillation modes as simple harmonic oscillations[11,15]. Here we proceed one more step to find dynamics of harmonic oscillator in the framework of GUP using Heisenberg picture of quantum mechanics. In Heisenberg picture of quantum mechanics, equation of motion for observable A is as follows,

$$\frac{dA}{dt} = \frac{i}{\hbar}[H, A]. \tag{6}$$

Hamiltonian for a simple harmonic oscillator is,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{7}$$

Now the equations of motion for x and p are respectively,

$$\frac{dx}{dt} = \frac{1}{m}\Big(p + \beta p^3\Big), \tag{8}$$

and

$$\frac{dp}{dt} = -\frac{1}{2}m\omega^2\Big(2x + \beta x p^2 + \beta p^2 x\Big). \tag{9}$$

Using Baker-Hausdorff lemma, a lengthy calculation gives the following equations for time evolution of x and p respectively,

$$x(t) = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t$$
$$+\beta\bigg[\frac{p^3(0)}{m\omega}(\omega t) - \frac{1}{2}\bigg(p(0)x(0)p(0) + \frac{3}{2}\Big[x(0)p^2(0) + p^2(0)x(0)\Big]\bigg)(\omega t)^2$$

$$-\left(\frac{5}{6}\frac{p^3(0)}{m\omega}-\frac{5}{12}m\omega\Big[x^2(0)p(0)+p(0)x^2(0)\Big]-\frac{1}{2}m\omega x(0)p(0)x(0)\right)(\omega t)^3$$

$$+\left(\frac{11}{24}\Big[x(0)p^2(0)+p^2(0)x(0)\Big]+\frac{5}{12}p(0)x(0)p(0)-\frac{1}{3}m^2\omega^2x^3(0)\right)(\omega t)^4\Bigg], \qquad (10)$$

and

$$p(t)=p(0)\cos\omega t-m\omega x(0)\sin\omega t$$

$$+\beta\Bigg[-\frac{1}{2}m\omega\Big[x(0)p^2(0)+p^2(0)x(0)\Big](\omega t)$$

$$-\left(p^3(0)-\frac{1}{4}m^2\omega^2\Big[p(0)x^2(0)+x^2(0)p(0)+2x(0)p(0)x(0)\Big]\right)(\omega t)^2$$

$$+\left(\frac{2}{3}m\omega\Big[x(0)p^2(0)+p^2(0)x(0)\Big]+\frac{1}{2}p(0)x(0)p(0)-\frac{1}{3}m^3\omega^3x^3(0)\right)(\omega t)^3\Bigg], \qquad (11)$$

where only terms proportional to first order of β are considered. It is evident that in the limit of $\beta\rightarrow 0$ one recover the usual results of ordinary quantum mechanics. The term proportional to β shows that in the framework of GUP harmonic oscillator is no longer "harmonic" essentially, since, now its time evolution has not oscillatory nature completely. In other words, in the framework of GUP there is no harmonic motion and this is a consequence of gravitational effect at quantum level.

Now for computing expectation values, we need a well-defined physical state. Note that eigenstates of position operators are not physical states because of existence of a minimal length which completely destroys the notion of locality. So we should consider a physical state such as $|\alpha\rangle$ where $|\alpha\rangle$ is for example a maximally localized or momentum space eigenstate[25]. Suppose that $p_\alpha(0)=\langle\alpha|p(0)|\alpha\rangle$ and $x_\alpha(0)=\langle\alpha|x(0)|\alpha\rangle$. Now the expectation value of momentum operator is,

$$\frac{\langle\alpha|p(t)|\alpha\rangle}{m}=\frac{p_\alpha(0)}{m}\cos\omega t-\omega x_\alpha(0)\sin\omega t$$

$$+\beta\Bigg[-\frac{1}{2}\omega\Big(x_\alpha(0)p_\alpha^2(0)+p_\alpha^2(0)x_\alpha(0)\Big)(\omega t)$$

$$-\left(\frac{p_\alpha^3(0)}{m}-\frac{1}{4}m\omega^2\Big[p_\alpha(0)x_\alpha^2(0)+x_\alpha^2(0)p_\alpha(0)+2x_\alpha(0)p_\alpha(0)x_\alpha(0)\Big]\right)(\omega t)^2$$

$$+\left(\frac{2}{3}\omega\Big[x_\alpha(0)p_\alpha^2(0)+p_\alpha^2(0)x_\alpha(0)\Big]+\frac{1}{2m}p_\alpha(0)x_\alpha(0)p_\alpha(0)-\frac{1}{3}m^2\omega^3x_\alpha^3(0)\right)(\omega t)^3\Bigg]. \qquad (12)$$

This relation shows that there is a complicated dependence of the expectation value of momentum operator to the mass of the oscillator. In usual quantum mechanics, $\frac{\langle\alpha|p(t)|\alpha\rangle}{m}$ and $\frac{p_\alpha(0)}{m}$ are mass independent. Here although $\frac{p_\alpha(0)}{m}$ is still mass independent, but now $\frac{\langle\alpha|p(t)|\alpha\rangle}{m}$ has a complicated mass dependence. This is a novel implication which have been induced by GUP. Physically, it is completely reasonable that the expectation value for momentum of a particle be a function of its mass, but the mass dependence here has a complicated form relative to usual situation.

4 Gup and Coherency

As a consequence of gravitational induced uncertainty, it seems that some basic notions such as coherency should be re-examined in this new framework. Here we want to show that in quantum gravity regime there is no coherent state at all. We consider the simple harmonic oscillator by Hamiltonian

$$H = \frac{1}{2m}(p^2 + m^2\omega^2 x^2) \tag{13}$$

The problem of quantum oscillator is easily solved in terms of the annihilation and creation operators a and $a^\dagger$. We recall the fundamental definitions:

$$a = \sqrt{\frac{m\omega}{2\hbar}}(x + \frac{ip}{m\omega}), \tag{14}$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(x - \frac{ip}{m\omega}) \tag{15}$$

and the inverse relations:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \qquad p = i\sqrt{\frac{m\hbar\omega}{2}}(-a + a^\dagger). \tag{16}$$

The Hamiltonian H is given in terms of these operators as :

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}) \tag{17}$$

If we set $N \equiv a^\dagger a$ (: Number operator),then

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \qquad [a^\dagger, a] = -1 \tag{18}$$

Let $\mathbf{H}$ be a Fock space generated by a and $a^\dagger$, and $\{|n\rangle | n \in \mathbf{\{}N\} \cup \{0\}\}$ be its basis. The action of a and $a^\dagger$ on $\mathbf{H}$ are given by

$$a|n\rangle = \sqrt{n}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \qquad N|n\rangle = n|n\rangle \tag{19}$$

Where $|0\rangle$ is a normalized vacuum ($a|0\rangle = 0$ and $\langle 0|0\rangle = 1$). Therefore states $|n\rangle$ for $n \geq 1$ are given by

$$|n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle. \tag{20}$$

These states satisfy the orthogonality and completeness conditions

$$\langle m|n\rangle = \delta_{mn}, \qquad \sum_{n=0}^{\infty} |n\rangle\langle n| = 1. \tag{21}$$

The coherent state was introduced by Schrödinger as the quantum state of the harmonic oscillator which minimizes the uncertainty equally distributed in both position x and momentum p. By definition, coherent state is the normalized state $|\lambda\rangle \in \mathbf{H}$, which is the eigenstate of annihilation operator and satisfies the following equation,

$$a|\lambda\rangle = \lambda|\lambda\rangle \qquad where \qquad \langle\lambda|\lambda\rangle = 1 \tag{22}$$

and

$$|\lambda\rangle = e^{-|\lambda|^2/2} \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}}|n\rangle = e^{-|\lambda|^2/2} e^{\lambda a^\dagger}|0\rangle. \tag{23}$$

Actually λ can be complex because a is not Hermittian. Let us now consider the following possibility, as a generalization for creation and annihilation operators in GUP,

$$a = \frac{1}{\sqrt{2\hbar\omega}}\Big(\omega x + i[p + f(p)]\Big), \tag{24}$$

$$a^\dagger = \frac{1}{\sqrt{2\hbar\omega}}\Big(\omega x - i[p + f(p)]\Big). \tag{25}$$

Here $f(p)$ is a function that satisfies three conditions, namely: (i) in the limit $\beta \to 0$ we recover the usual definition for the creation and annihilation operators,(14) and (15); (ii) if $\beta \neq 0$, then we have (3), and; (iii) $[a_{\vec{k}}, a^\dagger_{\vec{k'}}] = i\hbar\delta_{\vec{k}\vec{k'}}$, where $\vec{k}$ and $\vec{k'}$ are corresponding wave vectors. It can be shown that the following function satisfies the aforementioned restrictions

$$f(p_{\vec{k}}) = \sum_{n=1}^{\infty} \frac{(-\beta)^n}{2n+1} p_{\vec{k}}^{2n+1} \tag{26}$$

Condition (iii) means that the usual results, in relation with the structure of the Fock space, are valid in our case, for instance, the definition of the occupation number operator, $N_{\vec{k}} = a^\dagger_{\vec{k}} a_{\vec{k}}$, the interpretation of $a^\dagger_{\vec{k}}$ and $a_{\vec{k}}$ are creation and annihilation operators, respectively, etc. Clearly, the relation between $p_{\vec{k}}$, $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ is not linear, and from

the Hamiltonian (13) we now deduce that it is not diagonal in the occupation number representation. Let us now consider

$$f(p_{\vec{k}}) = -\frac{\beta}{3}p_{\vec{k}}^3 \tag{27}$$

In this form we find $p_{\vec{k}}$ as a function of $a_{\vec{k}}$ and $a_{\vec{k}}^{\dagger}$, namely

$$p_{\vec{k}} = -i\sqrt{\frac{\hbar\omega}{2}}(a_{\vec{k}} - a_{\vec{k}}^{\dagger})[1 - \sqrt{\frac{\hbar\omega\beta}{8}}(a_{\vec{k}} - a_{\vec{k}}^{\dagger})] \tag{28}$$

It is clear that, if $\beta = 0$ we recover the usual case. Rephrasing the Hamiltonian as a function of the creation and annihilation operators we find:

$$H = \sum_{\vec{k}} \hbar\omega[N_{\vec{k}} + \sqrt{\frac{\hbar\omega\beta}{8}}g(a_{\vec{k}}, a_{\vec{k}}^{\dagger}) + \beta\frac{(\hbar\omega)^2}{16}h(a_{\vec{k}}, a_{\vec{k}}^{\dagger})] \tag{29}$$

where functions $g(a_{\vec{k}}, a_{\vec{k}}^{\dagger})$ and $h(a_{\vec{k}}, a_{\vec{k}}^{\dagger})$ are:

$$g(a_{\vec{k}}, a_{\vec{k}}^{\dagger}) = a_{\vec{k}}^3 - N_{\vec{k}}a_{\vec{k}} - a_{\vec{k}}N_{\vec{k}} - a_{\vec{k}} - (a_{\vec{k}}^{\dagger})^3 + N_{\vec{k}}a_{\vec{k}}^{\dagger} + a_{\vec{k}}^{\dagger}N_{\vec{k}} + a_{\vec{k}}^{\dagger} \tag{30}$$

and

$$\begin{aligned} h(a_{\vec{k}}, a_{\vec{k}}^{\dagger}) &= a_{\vec{k}}^4 + a_{\vec{k}}^2(a_{\vec{k}}^{\dagger})^2 - a_{\vec{k}}^3a_{\vec{k}}^{\dagger} - a_{\vec{k}}^2a_{\vec{k}}^{\dagger}a_{\vec{k}} \\ &+(a_{\vec{k}}^{\dagger})^2a_{\vec{k}}^2 + (a_{\vec{k}}^{\dagger})^4 - (a_{\vec{k}}^{\dagger})^2a_{\vec{k}}a_{\vec{k}}^{\dagger} - (a_{\vec{k}}^{\dagger})^3a_{\vec{k}} \\ &-a_{\vec{k}}a_{\vec{k}}^{\dagger}a_{\vec{k}}^2 - a_{\vec{k}}(a_{\vec{k}}^{\dagger})^3 + a_{\vec{k}}a_{\vec{k}}^{\dagger}a_{\vec{k}}a_{\vec{k}}^{\dagger} + a_{\vec{k}}(a_{\vec{k}}^{\dagger})^2a_{\vec{k}} \\ &-a_{\vec{k}}^{\dagger}a_{\vec{k}}^3 - a_{\vec{k}}^{\dagger}a_{\vec{k}}(a_{\vec{k}}^{\dagger})^2 + a_{\vec{k}}^{\dagger}a_{\vec{k}}^2a_{\vec{k}}^{\dagger} + a_{\vec{k}}^{\dagger}a_{\vec{k}}a_{\vec{k}}^{\dagger}a_{\vec{k}}. \end{aligned} \tag{31}$$

Now with these pre-requisites we can consider the coherent states in the context of GUP. Suppose $|\lambda\rangle$ be an eigenstate of the annihilation operator. We remember that the definition of the annihilation operator in GUP may be different from the usual quantum mechanics but the fact that eigenstates of annihilation operator are coherent states do not changes. Therefore one can write

$$a|\lambda\rangle = \lambda|\lambda\rangle \tag{32}$$

Indeed $|n\rangle$ is the eigenstate of the number operator and satisfies completeness and orthogonality conditions. So we can expand $|\lambda\rangle$ in terms of the stationary states $|n\rangle$

$$|\lambda\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|\lambda\rangle = C_n|n\rangle, \tag{33}$$

The eigenvalue equation (19) implies the following recursion formula for the expansion coefficients:

$$C_n = \frac{\lambda}{\sqrt{n}} C_{n-1}. \tag{34}$$

We immediately obtain

$$C_n = \frac{\lambda^n}{\sqrt{n!}} C_0, \tag{35}$$

The constant C_0 is determined from the normalization condition on the Fock space,

$$1 = \langle \lambda | \lambda \rangle = |C_0|^2 \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{\sqrt{n!}} = |C_0|^2 e^{|\lambda|^2}, \tag{36}$$

For any complex number λ the correctly normalized quasi-classical state $|\lambda\rangle$ is therefore given by

$$|\lambda\rangle = e^{-\frac{1}{2}|\lambda|^2} \sum \frac{|\lambda|^n}{\sqrt{n!}} |n\rangle. \tag{37}$$

We recall that the n-th stationary state $|n\rangle$ is obtained from the ground state wave function by repeated application of the operator $a^\dagger$,

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle, \tag{38}$$

This allows us to write the coherent state in the form:

$$|\lambda\rangle = e^{-\frac{1}{2}|\lambda|^2} \sum_{n=0}^{\infty} \frac{1}{n!} (\lambda a^\dagger)^n |0\rangle = e^{-\frac{1}{2}|\lambda|^2} e^{\lambda a^\dagger} |0\rangle \tag{39}$$

We see that this expression for the eigenstates of the annihilation operator is the same as usual quantum mechanics, equation (23). Actually, it is not surprising that there is no changes in the form of states by modifying the uncertainty relation and similarly for the coherent state. The unchanged state itself cannot be the result of considering generalized uncertainty principle (GUP). It is because a quantum state does not necessarily imply a direct connection with uncertainty principle. Differences caused by different uncertainty relations (such as the GUP) will be found in the expectation values of the operators for a given state and their statistics(such as variance) that can be obtained from the measurement on the state. To analyze the coherent state under the GUP, we should consider $\langle x \rangle$ and $\langle p \rangle$ for the coherent state and see that whether they are changed or not. For this end, suppose that $|\lambda\rangle$ is a coherent state given by (39). Since

$$x = \sqrt{\frac{\hbar}{2\omega}} (a_{\vec{k}} + a_{\vec{k}}^\dagger), \tag{40}$$

and

$$p=-i\sqrt{\frac{\hbar\omega}{2}}(a_{\vec{k}}-a_{\vec{k}}^{\dagger})[1-\sqrt{\frac{\hbar\omega\beta}{8}}(a_{\vec{k}}-a_{\vec{k}}^{\dagger})], \tag{41}$$

one finds the following result for the expectation value of position operator, x

$$\langle x\rangle=\langle\lambda|x|\lambda\rangle=\sqrt{\frac{\hbar}{2\omega}}\langle\lambda|a_{\vec{k}}+a_{\vec{k}}^{\dagger}|\lambda\rangle=\sqrt{\frac{\hbar}{2\omega}}(\lambda+\lambda^*). \tag{42}$$

Therefore one has,

$$\langle x\rangle^2=\frac{\hbar}{2\omega}(\lambda^2+\lambda^{*2}+2\lambda\lambda^*)=\frac{\hbar}{2\omega}(\lambda+\lambda^*)^2. \tag{43}$$

It is straightforward to show that,

$$\langle x^2\rangle=\frac{\hbar}{2\omega}(\lambda^2+\lambda^{*2}+2\lambda\lambda^*+1)=\frac{\hbar}{2\omega}(\lambda+\lambda^*)^2+1, \tag{44}$$

and therefore we find for the variance of x,

$$(\Delta x)^2=\langle x^2\rangle-\langle x\rangle^2=\frac{\hbar}{2\omega}. \tag{45}$$

This is the same as usual quantum mechanics result. This is not surprising since the definition of position operator is the same as its definition in usual quantum mechanics. In the same manner, a simple calculation gives,

$$\langle p\rangle=-i\sqrt{\frac{\hbar\omega}{2}}\Big[(\lambda-\lambda^*)-\sqrt{\frac{\hbar\omega\beta}{8}}[(\lambda-\lambda^*)^2-1]\Big], \tag{46}$$

and

$$\langle p\rangle^2=-\frac{\hbar\omega}{2}\Big\{(\lambda-\lambda^*)^2-2\sqrt{\frac{\hbar\omega\beta}{8}}(\lambda-\lambda^*)[(\lambda-\lambda^*)^2-1]+$$

$$\frac{\hbar\omega\beta}{8}[(\lambda-\lambda^*)^2-1]^2\Big\}. \tag{47}$$

Since,

$$p^2=-\frac{\hbar\omega}{2}\Big[(a_{\vec{k}}-a_{\vec{k}}^{\dagger})^2-2\sqrt{\frac{\hbar\omega\beta}{8}}(a_{\vec{k}}-a_{\vec{k}}^{\dagger})^3+\frac{\hbar\omega\beta}{8}(a_{\vec{k}}-a_{\vec{k}}^{\dagger})^4\Big], \tag{48}$$

then,

$$\langle p^2\rangle=-\frac{\hbar\omega}{2}\Big\{[(\lambda-\lambda^*)^2-1]-$$

$$2\sqrt{\frac{\hbar\omega\beta}{8}}(\lambda^3-\lambda^{*3}-3\lambda^*\lambda^2+3\lambda^{*2}\lambda+3\lambda^*-3\lambda)+$$

$$\frac{\hbar\omega\beta}{8}(\lambda^4+\lambda^{*4}-4\lambda^*\lambda^3-4\lambda^{*3}\lambda+6\lambda^{*2}\lambda^2-6\lambda^2-6\lambda^{*2}+12\lambda^*\lambda+3)\Big\}, \qquad (49)$$

and therefore one finds,

$$(\Delta p)^2=\langle p^2\rangle-\langle p\rangle^2=-\frac{\hbar\omega}{2}[-1-2\sqrt{\frac{\hbar\omega\beta}{8}}(2\lambda^*-2\lambda)+\frac{\hbar\omega\beta}{8}(-4\lambda^2-4\lambda^{*2}+8\lambda^*\lambda+2)], \quad (50)$$

or by some manipulations, one obtains the following result for variance of p,

$$(\Delta p)^2=\frac{\hbar\omega}{2}+\hbar\omega\sqrt{\frac{\hbar\omega\beta}{2}}(\lambda^*-\lambda)+\frac{\hbar^2\omega^2\beta}{8}[1-2(\lambda^*-\lambda)^2] \qquad (51)$$

Note that these results give the usual quantum mechanical results when $\beta\longrightarrow 0$. Equations (45) and (51) show that although the definition of coherent states do not changes in GUP, but because of quantum gravitational effect expectation values and variances change considerably. Now product $(\Delta p)^2(\Delta x)^2$ has a complicated form which shows that complete coherency is impossible. Therefore, there is a considerable departure from very notion of coherency. In usual quantum mechanics one can have complete coherency in principle. One can localize wave packet in space completely, at least in principle, and wave can propagate without broadening, at least in principle. This is evident from $\Delta x\geq\frac{\hbar}{\Delta p}$. In quantum gravity because of gravitational induced uncertainty, one can not localize wave packet at all and it is impossible to cancel out broadening. Therefore in quantum gravity one can not have any solitonic states and any wave packet will suffer more broadening.

5 Wave Packet Propagation

The problem of wave packet propagation in quantum gravity first has been considered by Amelino-Camelia *et al*[27]. Using a κ-deformed Minkowski spacetime, they have investigated the experimental testability concerning the κ-deformed Minkowski relation between group velocity and momentum. Amelino-Camelia and Majid have considered the problem of waves propagation in Noncommutative Spacetime[28]. They have considered quantum group Fourier transform methods applied to the study of processes on noncommutative Minkowski spacetime. They have derived a wave equation and have investigated the associated phenomena of in vacuo dispersion. Assuming the deformation scale to be of the order of the Planck length they have found that the dispersion effects are large enough to be tested in experimental investigations of astrophysical phenomena such as gamma-ray bursts. Here in a simpler approach, we will show that there is an additional broadening

for wave packet due to gravitational induced uncertainty. This can be considered as a result of generalized dispersion relations or due to the variations in universal constants.

5.1 Wave Packet Propagation in Ordinary Quantum Mechanics

Consider the following plane wave profile,

$$f(x,t) \propto e^{ikx - i\omega t}. \tag{52}$$

Since $\omega = 2\pi\nu$, $k = \frac{2\pi}{\lambda}$ and $\nu = \frac{c}{\lambda}$, this equation can be written as $f(x,t) \propto e^{ik(x-ct)}$. Now the superposition of these plane waves with amplitude $g(k)$ can be written as,

$$f(x,t) = \int_{-\infty}^{\infty} dk\, g(k)\, e^{ik(x-ct)} = f(x-ct) \tag{53}$$

where $g(k)$ can have Gaussian profile. This wave packet is localized at $x - ct = 0$. In the absence dispersion properties for the medium, wave packet will not suffers any broadening with time. In this case the relation $\omega = kc$ holds. In general the medium has dispersion properties and therefore ω becomes a function of wave number, $\omega = \omega(k)$. In this situation equation (53) becomes,

$$f(x,t) = \int dk\, g(k)\, e^{ikx - i\omega(k)t}. \tag{54}$$

Suppose that $g(k) = e^{-\alpha(k-k_0)^2}$. With expansion of $\omega(k)$ around $k = k_0$, one find

$$\omega(k) \approx \omega(k_0) + (k-k_0)\Big(\frac{d\omega}{dk}\Big)_{k_0} + \frac{1}{2}(k-k_0)^2\Big(\frac{d^2\omega}{dk^2}\Big)_{k_0}, \tag{55}$$

where using the definitions,

$$\Big(\frac{d\omega}{dk}\Big)_{k_0} = v_g, \qquad \frac{1}{2}\Big(\frac{d^2\omega}{dk^2}\Big)_{k_0} = \mu, \qquad k - k_0 = k'. \tag{56}$$

equation (54) can be written as,

$$\begin{aligned} f(x,t) &= e^{ik_0x - i\omega(k_0)t}\int_{-\infty}^{\infty} dk'\, e^{-\alpha k'^2} e^{ik'(x-v_gt)}\, e^{-ik'^2\beta t} \\ &= e^{ik_0x - i\omega(k_0)t}\int_{-\infty}^{\infty} dk'\, e^{ik'(x-v_gt)}\, e^{-(\alpha+i\mu t)k'^2}. \end{aligned} \tag{57}$$

Now completing the square root in exponent and integration gives,

$$f(x,t) = e^{i\left[k_0x - \omega(k_0)t\right]}\left(\frac{\pi}{\alpha + i\mu t}\right)^{\frac{1}{2}} e^{-\left[\frac{(x-v_gt)^2}{4(\alpha+i\mu t)}\right]}. \tag{58}$$

Therefore one find,

$$|f(x,t)|^2 = \left(\frac{\pi^2}{\alpha^2+\mu^2t^2}\right)^{\frac{1}{2}} e^{-\left[\frac{\alpha(x-v_g t)^2}{2(\alpha^2+\mu^2t^2)}\right]}, \tag{59}$$

which is the profile of the wave in position space. The quantity which in $t = 0$ was α, now has became $\alpha + \frac{\mu^2t^2}{\alpha}$ and this is the notion of broadening. Therefore,

$$Broadening \propto \left(1+\frac{\mu^2t^2}{\alpha^2}\right)^{\frac{1}{2}}. \tag{60}$$

This relation shows that a wave packet with width $(\Delta x)_0$ in $t = 0$ after propagation will have the following width,

$$(\Delta x)_t = (\Delta x)_0\left(1+\frac{\mu^2t^2}{\alpha^2}\right)^{\frac{1}{2}}. \tag{61}$$

5.2 Wave Packet Propagation in Quantum Gravity

As has been indicated, when one considers gravitational effects, usual uncertainty relation of Heisenberg should be replaced by,

$$\Delta x \geq \frac{\hbar}{\Delta p} + \frac{\alpha' l_p^2 \Delta p}{\hbar}. \tag{62}$$

As a first step analysis we consider the above simple form of GUP. Suppose that

$$\Delta x \sim x, \qquad \Delta p \sim p, \qquad p = \hbar k, \qquad x = \bar{\lambda} = \frac{\lambda}{2\pi}.$$

Therefore one can write,

$$\bar{\lambda} = \frac{1}{k} + \alpha' l_p^2\, k \qquad and \qquad \omega = \frac{c}{\bar{\lambda}}. \tag{63}$$

In this situation the dispersion relation becomes,

$$\omega = \omega(k) = \frac{kc}{1+\alpha' l_p^2\, k^2}. \tag{64}$$

This relation can be described in another viewpoint. By expansion of $\left(1+\alpha' l_p^2\, k^2\right)^{-1}$ and neglecting second and higher order terms of α', we find that $\omega = kc(1-\alpha' l_p^2\, k^2)$. This can be considered as $\omega = k'c$ where $k' = k(1-\alpha' l_p^2\, k^2)$. Now one can define a generalized momentum as $p = \hbar k' = \hbar k(1-\alpha' l_p^2\, k^2)$. It is possible to consider this equation as $p = \hbar' k$

where $\hbar' = \hbar(1-\alpha' l_p^2 k^2)$. So one can interpret it as a wave number dependent Planck "constant". In the same manner group velocity becomes,

$$v_g = \frac{d\omega}{dk}\Big|_{k=k_0} = \frac{c(1-\alpha' l_p^2 k^2)}{(1+\alpha' l_p^2 k^2)^2}\Big|_{k=k_0}. \tag{65}$$

Up to first order in α' this relation reduces to $v_g \approx c(1-3\alpha' l_p^2 k_0^2)$.
A little algebra gives μ as follow

$$\mu = \frac{1}{2}\Big(\frac{d^2\omega}{dk^2}\Big)\Big|_{k=k_0} = \frac{-3\alpha' l_p^2 c\, k(1+\alpha' l_p^2 k^2)^2 + 4\alpha'^2 l_p^4 c\, k^3(1+\alpha' l_p^2 k^2)}{(1+\alpha' l_p^2 k^2)^4}\Bigg|_{k=k_0}, \tag{66}$$

which up to first order in α' reduces to $\mu \approx -3\alpha' l_p^2 c k_0$. It is evident that when $\alpha' \to 0$ then $\mu \to 0$ and $v_g \to c$. The same analysis which has leads us to equation (61), now gives the following result,

$$(\Delta x)_t = (\Delta x)_0\Bigg(1+\frac{1}{\alpha^2}\Big(\frac{-3\alpha' l_p^2 c\, k_0(1+\alpha' l_p^2 k_0^2)^2 + 4\alpha'^2 l_p^4 c\, k_0^3(1+\alpha' l_p^2 k_0^2)}{(1+\alpha' l_p^2 k_0^2)^4}\Big)^2 t^2\Bigg)^{\frac{1}{2}}. \tag{67}$$

If one accepts that α' is negative constant ($\alpha' < 0$), then group velocity of the wave packet becomes greater than light velocity. This is evident from equation (65) and is reasonable from varying speed of light models. In fact if $|\alpha'|k^2 l_p^2 \ll 1$, one recover usual quantum mechanics but when $|\alpha'|k^2 l_p^2 \approx 1$, Planck scale quantum mechanics will be achieved. Based on this argument, equation (67) shows that in quantum gravity there exists a more broadening of wave packet due to gravitational effects. Up to first order in α', this equation becomes,

$$(\Delta x)_t = (\Delta x)_0\Bigg(1-\frac{3\alpha' l_p^2 c\, k_0 t^2}{\alpha^2}\Bigg)^{\frac{1}{2}}. \tag{68}$$

Now using equation (64), one can write the generalized dispersion relation as the following form also,

$$\omega(p) = \frac{\hbar p c}{\hbar^2 + \alpha' l_p^2 p^2}, \tag{69}$$

or

$$E' = \hbar\omega(p) = \frac{pc}{1+\alpha'\big(\frac{l_p p}{\hbar}\big)^2}. \tag{70}$$

It is evident that if $\alpha' \longrightarrow 0$ Then $E' \longrightarrow E = pc$ and $\omega(p) \longrightarrow \omega = \frac{pc}{\hbar}$.

6 Summary

In this paper the effect of gravitation on some well-known quantum optical phenomena has been studied. Considering dynamics and quantum mechanical coherent states of a simple harmonic oscillator in the framework of Generalized Uncertainty Principle(GUP), we have derived the equation of motion for simple harmonic oscillator and some of their new implications have been discussed. As an important consequence we have shown that essentially, there is no harmonic oscillation in quantum gravity regime. Then coherent states of harmonic oscillator in the case of GUP are compared with relative situation in ordinary quantum mechanics. It is shown that in the framework of GUP there is no considerable difference in definition of coherent states relative to ordinary quantum mechanics. But, considering expectation values and variance of some operators, based on quantum gravitational arguments one concludes that although it is possible to have complete coherency and vanishing broadening in usual quantum mechanics, gravitational induced uncertainty destroys complete coherency in quantum gravity and it is impossible to have a monochromatic ray in principle. Finally we have shown that there is an extra broadening in wave packet propagation due to quantum gravitational effects. This leads us to generalized dispersion relation. Generalized dispersion relations can be described as a possible framework for varying constant of the nature. Since quantum gravitational effects are very small, their possible detection requires very high energy experiments. It seems that LHC will provide a reasonable framework for testing these predictions.

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